

EXAMPLES OF TYPE III_1 INHOMOGENOUS MARKOV SHIFTS SUPPORTED ON TOPOLOGICAL MARKOV SHIFTS

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ABSTRACT. We construct inhomogenous Markov measures for which the shift is of Kreiger type III_1 . These measures are fully supported on a topological Markov shift space of the hyperbolic toral automorphism $f(x, y) = (\{x + y\}, x) : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ and are used in a subsequent paper [Kos1] in the construction of C^1 Anosov diffeomorphisms of \mathbb{T}^2 without an absolutely continuous invariant measure.

1. INTRODUCTION

Topological markov shifts (TMS) are an important model in ergodic theory which plays a central role in smooth dynamics of $C^{1+\alpha}$ hyperbolic diffeomorphisms. Indeed, the construction of Markov partitions of Adler and Weiss [AW] (toral automorphisms), Sinai [Si] (Anosov diffeomorphisms) and Bowen [B] (Axion A diffeomorphisms) shows that given an hyperbolic diffeomorphism there exists a TMS and a finite to one factor map from the TMS Σ to the manifold M which intertwines the actions of the shift T and the diffeomorphism f . This factor map and the symbolic model were used extensively in smooth dynamics, examples include showing existence and uniqueness of the measure of maximal entropy and equilibrium states, proving central limit theorems, Livsic' type theorems and more.

Another feature of the factor map $\pi : \Sigma \rightarrow M$ is that it is finite to one except on the (volume) measure zero set which consists of the union of iterations of f on the (topological) boundary of the atoms of the Markov partition, see example 1 below. This shows that a large class of measures on M for which f is non singular (also known as quasi invariant), which includes all the measures for which f satisfies Poincare recurrence, can be lifted back (since π^{-1} is a function on the support of the measure) to the TMS. This paper deals with constructions of inhomogenous markov measures on TMS for which the shift is non singular, conservative (recurrent) and ergodic yet there exists no absolutely continuous shift invariant measure. In fact the system $(\Sigma, \mathcal{B}_\Sigma, \mu, T)$ is of Krieger type III_1 . These examples are Markovian analogues of the Bernoulli shifts with non identical factor measures as in [Ham, Kre, Kos].

The main motivation for us in this work is the paper [Kos1] where these measures are used in the construction of a conservative, ergodic C^1 Anosov diffeomorphism of \mathbb{T}^2 without a Lebesgue absolutely continuous invariant measure. These diffeomorphisms are obtained by smooth realization of $(\Sigma, \mathcal{B}_\Sigma, \mu, T)$.

This paper is organised as follows. Section 2 contains the basic definitions from ergodic theory and the probability theory of Markov chains which will be used in this paper. Section 3 starts with a discussion on the condition of non singularity for the shift map with respect to a half stationary inhomogenous Markov measure and a sufficient condition for exactness of the one sided shift. After this we present the construction of the measures and prove that the shift is of type III_1 with respect to these measures. We end the paper with an explanation on how to construct such measures on a general TMS, some remarks about these measures and an open question on the possible Krieger types of Markov shifts.

2. PRELIMINARY DEFINITIONS

2.1. Short introduction to non singular ergodic theory. For more details and explanations we refer the reader to [Aar1].

The research of Z.K. was supported in part by the European Advanced Grant StochExtHomog (ERC AdG 320977).

Let (X, \mathcal{B}, μ) be a standard probability space. In what follows equalities (and inclusions) of sets are modulo the measure μ on the space. A measurable map $T : X \rightarrow X$ is *non singular* if $T_*\mu := \mu \circ T^{-1}$ is equivalent to μ meaning that they have the same collection of negligible sets. If T is invertible one has the Radon Nykodym derivatives

$$(T^n)'(x) := \frac{d\mu \circ T^n}{d\mu}(x) : X \rightarrow \mathbb{R}_+,$$

A set $W \subset X$ is *wandering* if $\{T^n W\}_{n \in \mathbb{Z}}$ are pairwise disjoint and T is *conservative* if there exists no wandering set of positive measure. By Hopf's criteria, an invertible T is conservative (w.r.t μ) if and only if

$$\sum_{n=1}^{\infty} (T^n)'(x) = \infty \quad \mu - a.e.$$

A transformation T is *ergodic* if there are no non trivial T invariant sets. That is $T^{-1}A = A$ implies $A \in \{\emptyset, X\}$. If (X, \mathcal{B}, μ) is a non atomic measure space and T is invertible and ergodic then T is also conservative. The converse implication is not true in general as there are conservative non-ergodic transformations. Since proving ergodicity is usually a harder task then proving conservativity we would like to concentrate on a class of transformations, called K -automorphisms, for which conservativity implies ergodicity.

A transformation is a *K-automorphism* if there exists a σ -algebra $\mathcal{F} \subset \mathcal{B}$ such that:

- $T^{-1}\mathcal{F} \subset \mathcal{B}$ meaning that \mathcal{F} is a factor of \mathcal{B} .
- $\cap_{n=1}^{\infty} T^{-n}\mathcal{F} = \{\emptyset, X\}$ (exactness) and $\vee_{n \in \mathbb{Z}} T^n\mathcal{F} = \mathcal{B}$ (exhaustiveness).
- $T'(x)$ is \mathcal{F} measurable.

The first two properties are the standard definition of a K -automorphism in the case of measure preserving automorphisms. The condition on the measurability of the Radon Nykodym derivative comes to ensure that (X, \mathcal{B}, μ, T) is the **unique** natural extension of the non invertible exact transformation $(X/\mathcal{F}, \mathcal{F}, \mu|_{\mathcal{F}}, T)$. It was shown in [Kre, ST] that a conservative and K transformation is necessarily ergodic.

The *Krieger ratio set* $R(T)$: We say that $r \geq 0$ is in $R(T)$ if for every $A \in \mathcal{B}$ of positive μ measure and for every $\epsilon > 0$ there exists an $n \in \mathbb{Z}$ such that

$$\mu(A \cap T^{-n}A \cap \{x \in X : |(T^n)'(x) - r| < \epsilon\}) > 0.$$

The ratio set of an ergodic measure preserving transformation is a closed multiplicative subgroup of $[0, \infty)$ and hence it is of the form $\{0\}, \{1\}, \{0, 1\}, \{0\} \cap \{\lambda^n : n \in \mathbb{Z}\}$ for $0 < \lambda < 1$ or $[0, \infty)$. Several ergodic theoretic properties can be seen from the ratio set. One of them is that $0 \in R(T)$ if and only if there exists no σ -finite T -invariant μ - a.c.i.m. Another interesting relation is that $1 \in R(T)$ if and only if T is conservative (Maharam's Theorem). If $R(T) = [0, \infty)$ we say that T is of type III₁.

2.1.1. *Topological Markov shifts.* A *topological Markov shift* (TMS) on S is the shift on a shift invariant subset $\Sigma \subset S^{\mathbb{Z}}$ of the form

$$\Sigma_A := \left\{ x \in S^{\mathbb{Z}} : A_{x_i, x_{i+1}} = 1 \right\},$$

where $A = \{A_{s,t}\}_{s,t \in S}$ is a $\{0, 1\}$ valued matrix on S . A TMS is mixing if there exists $n \in \mathbb{N}$ such that $A_{s,t}^n > 0$ for every $s, t \in S$.

TMS appear in ergodic theory as a symbolic model for $C^{1+\alpha}$ Anosov and Axiom A diffeomorphisms via the construction Markov partitions of the manifold M [AW, Si, B, Adl]. We present here an example which motivates the choice of the TMS we will be performing the construction on.

Example 1. Consider $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ the Toral automorphism defined by

$$f(x, y) = (\{x + y\}, x) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \mod 1,$$

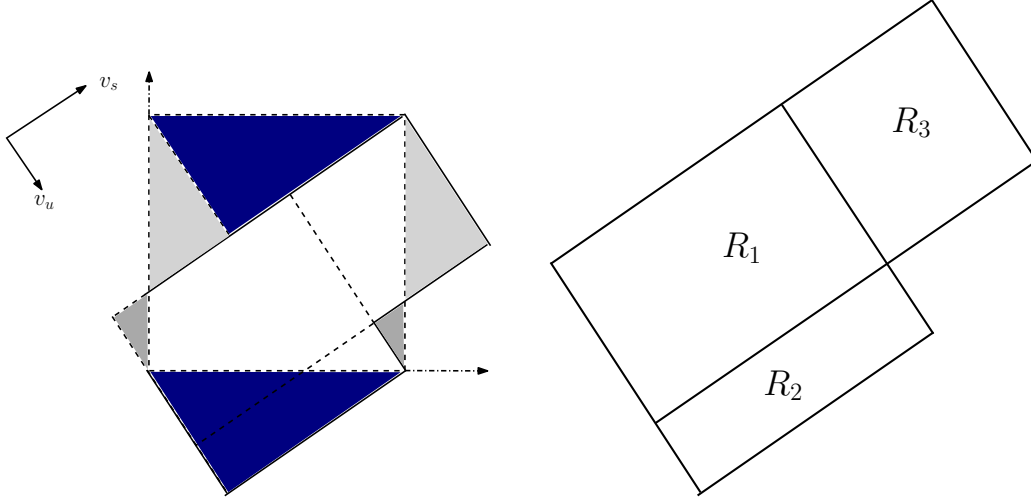


FIGURE 2.1. The construction of the Markov partition

where $\{t\}$ is the fractional part of t . Since $\left| \det \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right| = 1$, f preserves the Lebesgue measure on \mathbb{T}^2 . In [AW], a method is given for the construction of Markov partitions for hyperbolic toral automorphisms. For this example one has a Markov partition with three elements $\{R_1, R_2, R_3\}$, see figure 2.1 and [Adl].

The adjacency Matrix of the Markov partition is then defined by $A_{i,j} = 1$ if and only if $R_i \cap f^{-1}(R_j) \neq \emptyset$. In this example,

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Let $\Phi : \Sigma_{\mathbf{A}} \rightarrow \mathbb{T}^2$ be the map defined by

$$\Phi(x) := \bigcap_{n=-\infty}^{\infty} \overline{f^{-n}R_{x_n}}.$$

Note that $\left\{ \bigcap_{n=-N}^N \overline{f^{-n}R_{x_n}} \right\}_{N=1}^{\infty}$ is a decreasing sequence of compact sets, hence by the Baire Category Theorem, $\Phi(x)$ is well defined. The map $\Phi : \Sigma_{\mathbf{A}} \rightarrow \mathbb{T}^2$ is continuous, finite to one, and for every $x \in \Sigma_{\mathbf{A}}$,

$$\Phi \circ T(x) = f \circ \Phi(x).$$

Thus Φ is a semi-conjugacy (topological factor map) between $(\Sigma_{\mathbf{A}}, T)$ to (\mathbb{T}^2, f) . In addition, for every $x \in \mathbb{T}^2 \setminus \bigcup_{n \in \mathbb{Z}} \bigcup_{i=1}^3 f^{-n}(\partial R_i)$ there exists a unique $w \in \Sigma_{\mathbf{A}}$ so that $w = \Phi^{-1}(x)$. The Lebesgue measure λ on \mathbb{T}^2 is f invariant and $\lambda\left(\bigcup_{n \in \mathbb{Z}} \bigcup_{i=1}^3 f^{-n}(\partial R_i)\right) = 0$. Thus Φ defines an isomorphism between $(\mathbb{T}^2, \lambda, f)$ and $(\Sigma_{\mathbf{A}}, \mu_{\pi_{\mathbf{Q}}, \mathbf{Q}}, T)$ where $\mu_{\pi_{\mathbf{Q}}, \mathbf{Q}} = \Phi_* \lambda$ is the stationary Markov measure with

$$(2.1) \quad P_j \equiv \mathbf{Q} := \begin{pmatrix} \frac{\varphi}{1+\varphi} & 0 & \frac{1}{1+\varphi} \\ \frac{\varphi}{1+\varphi} & 0 & \frac{1}{1+\varphi} \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$(2.2) \quad \pi_j = \pi_{\mathbf{Q}} := \begin{pmatrix} 1/\sqrt{5} \\ 1/\varphi\sqrt{5} \\ 1/\varphi\sqrt{5} \end{pmatrix} = \begin{pmatrix} \lambda(R_1) \\ \lambda(R_2) \\ \lambda(R_3) \end{pmatrix}.$$

2.2. Markov Chains.

2.2.1. *Basics of Stationary (homogenous) Chains.* Let S be a finite set which we regard as the state space of the chain, $\pi = \{\pi(s)\}_{s \in S}$ a probability vector on S and $P = (P_{s,t})_{s,t \in S}$ a stochastic matrix. The vector π and P define a Markov chain $\{X_n\}$ on S by

$$\begin{aligned}\mathbb{P}_\pi(X_0 = t) &:= \pi(t) \\ \mathbb{P}(X_n = s | X_1, \dots, X_{n-1}) &:= P_{X_{n-1}, s}.\end{aligned}$$

P is *irreducible* if for every $s, t \in S$, there exists $n \in \mathbb{N}$ such that

$$P_{s,t}^n = \mathbb{P}(X_n = t | X_0 = s) > 0,$$

and P is *aperiodic* if for every $s \in S$

$$\gcd\{n : P_{s,s}^n > 0\} = 1.$$

Given an irreducible and aperiodic P , there exists a unique stationary $(\pi_P P = \pi_P)$ probability vector π_P . In addition for every $s, t \in S$,

$$P_{s,t}^n \xrightarrow{n \rightarrow \infty} \pi_P(t).$$

Since S is a finite state space, it follows that for any initial distribution π on S ,

$$\mathbb{P}_\pi(X_n = t) = \sum_{s \in S} \pi(s) P_{s,t}^n \xrightarrow{n \rightarrow \infty} \pi_P(t).$$

An important fact which will be used in the sequel is that the stationary distribution is continuous with respect to the stochastic matrix. That is if $\{P_n\}_{n=1}^\infty$ is a sequence of irreducible and aperiodic stochastic matrices such that

$$\|P_n - P\|_\infty := \max_{s,t \in S} |(P_n)_{s,t} - P_{s,t}| \xrightarrow{n \rightarrow \infty} 0$$

and P is irreducible and aperiodic then

$$\|\pi_{P_n} - \pi_P\|_\infty \rightarrow 0.$$

2.2.2. *Non singular Markov shifts:* Let S be a finite set. An inhomogeneous Markov Chain is a stochastic process $\{X_n\}_{n \in \mathbb{Z}}$ such that for each times $t_1, \dots, t_l \in \mathbb{Z}$, and $s_1, \dots, s_l \in S$,

$$\mathbb{P}(X_{t_1} = s_1, X_{t_2} = s_2, \dots, X_{t_l} = s_l) = \mathbb{P}(X_{t_1} = s_1) \prod_{k=1}^{l-1} \mathbb{P}(X_{t_{k+1}} = s_{k+1} | X_{t_k} = s_k).$$

Note that unlike in the classical setting of Markov Chains $\mathbb{P}(X_{t_{k+1}} = s_{k+1} | X_{t_k} = s_k)$ can depend on t_k .

The ergodic theoretical formulation is as follows. Let $\{P_n\}_{n=-\infty}^\infty \subset M_{S \times S}$ be a sequence of stochastic matrices on S . In addition let $\{\pi_n\}_{n=-\infty}^\infty$ be a sequence of probability distributions on S so that for every $s \in S$ and $n \in \mathbb{Z}$,

$$(2.3) \quad \sum_{t \in S} \pi_{n-1}(t) \cdot P_n(t, s) = \pi_n(s).$$

Then one can define a measure on the collection of cylinder sets,

$$[b]_k^l := \left\{ x \in S^{\mathbb{Z}} : x_j = b_j \ \forall j \in [k, l] \cap \mathbb{Z} \right\}$$

by

$$\mu([b]_k^l) := \pi_k(b_k) \prod_{j=k}^{l-1} P_j(b_j, b_{j+1}).$$

Since the equation (2.3) is satisfied, μ satisfies the consistency condition. Therefore by Kolmogorov's extension theorem μ defines a measure on $S^{\mathbb{Z}}$. In this case we say that μ is the Markov measure generated by $\{\pi_n, P_n\}_{n \in \mathbb{Z}}$ and denote $\mu = M\{\pi_n, P_n : n \in \mathbb{Z}\}$. By $M\{\pi, P\}$ we mean the measure generated by $P_n \equiv P$ and $\pi_n \equiv \pi$. We say that μ is non singular for the shift T on $S^{\mathbb{Z}}$ if $T_*\mu \sim \mu$. See subsection

3.0.3 for an extension of Kakutani's Theorem for product measures to a class of inhomogenous Markov measures which is used in giving a criteria for non singularity of the shift in our examples.

3. TYPE III₁ MARKOV SHIFTS SUPPORTED ON TOPOLOGICAL MARKOV SHIFTS

This section is organized as follows. First we give a condition for non singularity of the Markov measure which is an application of ideas of Cabanos, Liptzer and Shiryaev [Shi, LM] and prove a necessary condition for exactness of a one sided shift. Then we construct the aforementioned examples and prove that they are type III₁ measures for the shift.

3.0.3. Non Singularity criteria for Markov shifts. In order to check if a measure is shift non singular we apply the following reasoning of [Shi], see also [LM].

Definition 2. Given a filtration $\{\mathcal{F}_n\}$, we say that $\nu \ll^{loc} \mu$ (ν is locally absolutely continuous with respect to μ) if for every $n \in \mathbb{N}$

$$\nu_n \ll \mu_n$$

where

$$\nu_n = \nu|_{\mathcal{F}_n}.$$

Suppose that $\nu \ll^{loc} \mu$ w.r.t $\{\mathcal{F}_n\}$, set

$$z_n := \frac{d\nu_n}{d\mu_n},$$

and

$$\alpha_n(x) := z_n(x) \cdot z_{n-1}^\oplus(x),$$

where $z_{n-1}^\oplus = \frac{1}{z_{n-1}} \cdot \mathbf{1}_{[z_{n-1} \neq 0]}$. The question is when $\nu \ll^{loc} \mu$ implies $\nu \ll \mu$.

Theorem 3. [Shi, Thm. 4, p. 528]. *If $\nu \ll^{loc} \mu$ then $\nu \ll \mu$ if and only if*

$$\sum_{k=1}^{\infty} [1 - E_\mu(\sqrt{\alpha_n} | \mathcal{F}_{n-1})] < \infty \quad \nu \text{ a.s.}$$

If $\nu \ll \mu$ then

$$\frac{d\nu}{d\mu} = \lim_{n \rightarrow \infty} z_n.$$

Given a a markovian measure $\mu = M\{\pi_n, P_n : n \in \mathbb{Z}\}$ on $S^{\mathbb{Z}}$, we want to know when $\mu \sim \mu \circ T$. The natural filtration on the product space is the sequence of algebras $\mathcal{F}_n := \sigma\{[b]_{-n}^n; b \in S^{\mathbb{Z} \cap [-n, n]}\}$.

The measures which we will construct are fully supported on a TMS $\Sigma_{\mathbf{A}}$ meaning that for every $n \in \mathbb{N}$,

$$\text{supp} P_n := \{(s, t) \in S \times S : P_n(s, t) > 0\} = \text{supp} \mathbf{A}.$$

This implies that $\mu \circ T \ll^{loc} \mu$. In addition the measure $M\{P_n, \pi_n\}$ will be half stationary in the sense that for every $j \leq 0$, $P_j := \mathbf{Q}$ where \mathbf{Q} is the matrix from Example 1. This condition implies that for every $j \leq 0$, $\pi_j = \pi_{\mathbf{Q}}$ and

$$\alpha_n(x) = \frac{P_{n-1}(x_{n-1}, x_n)}{P_n(x_{n-1}, x_n)}, \quad \forall n > 0.$$

By Theorem 3, in this setting, μ -non singular if and only if

$$\sum_{n=-\infty}^{\infty} [1 - E_\nu(\sqrt{\alpha_n} | \mathcal{F}_{n-1})(x)] = \sum_{n=0}^{\infty} \left[1 - \sum_{s \in S} \sqrt{P_{n-1}(x_{n-1}, s) P_n(x_{n-1}, s)} \right] < \infty$$

for $\mu \circ T$ a.e. x . The following corollary concludes our discussion.

Corollary 4. *Let $\nu = M\{\pi_n, P_n : n \in \mathbb{Z}\}$, where $\{P_n\}$ are fully supported on a TMS $\Sigma_{\mathbf{A}}$ and there exists an aperiodic and irreducible $P \in M_{S \times S}$ such that for all $n \leq 0$, $P_n \equiv P$*

- $\nu \circ T \sim \nu$ if and only if

$$(3.1) \quad \sum_{n=0}^{\infty} \sum_{s \in S} \left(\sqrt{P_n(x_n, s)} - \sqrt{P_{n-1}(x_n, s)} \right)^2 < \infty, \quad \nu \circ T \text{ a.s. } x.$$

- If $\nu \circ T \sim \nu$ then for all $n \in \mathbb{N}$,

$$(T^n)'(x) = \prod_{k=0}^{\infty} \frac{P_{k-n}(x_k, x_{k+1})}{P_k(x_k, x_{k+1})}.$$

A condition for exactness of the one sided shift. Let S be a countable set and $\{(\pi_n, P_n)\}_{n=1}^{\infty} \subset \mathcal{P}(S) \times \mathcal{M}_{S \times S}$. Denote the one sided shift on $S^{\mathbb{N}}$ by σ and by \mathcal{F} the Borel σ -algebra of $S^{\mathbb{N}}$. The following is a sufficient condition for exactness (trivial tail σ -field) of the one sided shift which is well known in the theory of non homogenous Markov chains. We include a simple ergodic theoretic proof for the sake of completeness.

Proposition 5. *Let S be a countable set and μ be a Markovian measure on $S^{\mathbb{N}}$ which is defined by $\{\pi^{(k)}, P^{(k)} : k \in \mathbb{N} \cup \{0\}\}$. If there exists $C > 0$ and $N_0 \in \mathbb{N}$ so that for every $s, t \in S$, and $k \in \mathbb{N}$,*

$$(3.2) \quad (P_k P_{k+1} \cdots P_{k+N_0-1})_{s,t} \geq C$$

then the one sided shift $(S^{\mathbb{N} \cup \{0\}}, \mathcal{F}, \mu, \sigma)$ is exact.

Remark. In the setting of Markov maps, exactness was proved under various distortion properties [see [Aar1, Th]]. Their conditions guarantees the existence of an absolutely continuous σ -finite invariant measure.

Proof. The measure $\mu \circ T^{-n}$ is the Markov measure generated by $Q_k := P_{k+n}$ and $\tilde{\pi}_k := \pi_{k+n}$. Let α_n be the collection of n cylinders of the form $[d]_1^n$ and $\alpha^* = \cup_n \alpha_n$.

For every $D = [a]_0^n \in \alpha_n$ and $B = [b]_0^{n(B)} \in \alpha^*$,

$$\begin{aligned} \mu \left(D \cap T^{-(n+N_0)} B \right) &= \mu(D) (P_n P_{n+1} \cdots P_{n+N_0-1})_{a_n, b_0} \prod_{j=0}^{n(B)-1} P_{N_0+n+j}(b_j, b_{j+1}) \\ &\geq C \mu(D) \pi_{n+N_0}(b_0) \prod_{j=0}^{n(B)-1} P_{N_0+n+j}(b_j, b_{j+1}) = C \cdot \mu(D) \mu \circ T^{-(n+N_0)}(B). \end{aligned}$$

Consequently for all $B \in \mathcal{F}$ and $D \in \alpha_n$,

$$\mu \left(D \cap T^{-(n+N_0)} B \right) \geq C \cdot \mu(D) \mu \circ T^{-(n+N_0)}(B).$$

Let $B \in \cap_{n=1}^{\infty} \sigma^{-n} \mathcal{F}$ and $D \in \alpha_n$. Writing $B_{n+N_0} \in \mathcal{F}$ for a set such that $B = T^{-n-N_0} B_{n+N_0}$,

$$\begin{aligned} \mu(D \cap B) &= \mu \left(D \cap T^{-(n+N_0)} B_{n+N_0} \right) \\ &\geq C \cdot \mu(D) \mu \circ T^{-(n+N_0)}(B_{n+N_0}) \\ &= C \mu(D) \mu(B) \end{aligned}$$

and thus for every $n \in \mathbb{N}$, $\mu(B | \alpha_n) \geq C \mu(B)$. Since $\alpha_n \uparrow \alpha^*$ and α^* generates \mathcal{F} ,

$$\mu(B | \alpha_n)(x) \xrightarrow{n \rightarrow \infty} 1_B(x) \quad \mu - a.s.$$

by the Martingale convergence theorem. It follows that if $\mu(B) > 0$ then

$$1_B(x) \geq C \mu(B) > 0 \quad \mu - a.s.$$

This shows that for every $B \in \cap_{n=1}^{\infty} \sigma^{-n} \mathcal{F}$, $\mu(B) \in \{0, 1\}$ (the shift is exact). □

3.1. Type III₁ Markov Shifts. We will construct a Markov measure supported on $\Sigma_{\mathbf{A}}$ from Example 1. In the end of this subsection we will explain what needs to be altered in the case of a general mixing TMS and conclude with some open questions.

In this subsection, let $\Omega := \Sigma_{\mathbf{A}}$, $\mathcal{B} := \mathcal{B}_{\Sigma_{\mathbf{A}}}$ and T is the two sided shift on Ω . For two integers $k < l$, write $\mathcal{F}(k, l)$ for the algebra of sets generated by cylinders of the form $[b]_k^l$, $b \in \{1, 2, 3\}^{l-k}$.

3.1.1. Idea of the construction of the type III Markov measure. The construction uses the ideas in [Kos1]. For every $j \leq 0$

$$P_j \equiv \mathbf{Q} \text{ and } \pi_j \equiv \pi_{\mathbf{Q}},$$

where \mathbf{Q} and $\pi_{\mathbf{Q}}$ are as in (2.1) and (2.2) respectively. On the positive axis one defines on larger and larger chunks the stochastic matrices which depend on a distortion parameter $\lambda_k \geq 1$ where 1 means no distortion. Now a cylinder set $[b]_{-n}^n$ fixes the values of the first n terms in the product form of the Radon Nykodym derivatives. As a natural first step in proving that a given number is in the ratio set is to check the condition for cylinder sets, we would like to be able to correct the values of the Radon Nykodym derivatives which were fixed by the cylinder set. This corresponds to a lattice condition on λ_k which is less straightforward then the one in [Kos]. The measure of the set $[b]_{-n}^n \cap T^{-N}[b]_{-n}^n \cap \left\{ (T^N)' \approx a \right\}$ could be very small which forces us to look for many approximately independent such events so that their union covers at least a fixed proportion of $[b]_{-n}^n$. In the product measure case we used independence of the coordinates. For a Markov measure the coordinates are not independent and this leads us to a condition which comes from the convergence to the stationary distribution and the mixing property for stationary chains.

More specifically the construction goes as follows. We define inductively 5 sequences $\{\lambda_j\}$, $\{m_j\}$, $\{n_j\}$, $\{N_j\}$ and $\{M_j\}$ where

$$\begin{aligned} M_0 &= 1 \\ N_j &:= N_{j-1} + n_j \\ M_j &:= N_j + m_j. \end{aligned}$$

This defines a partition of \mathbb{N} into segments $\{[M_{j-1}, N_j), [N_j, M_j)\}_{j=1}^{\infty}$. The sequence $\{P_n\}$ equals \mathbf{Q} on the $[N_j, M_j)$ segments while on the $[M_{j-1}, N_j)$ segments we have $P_n \equiv \mathbf{Q}_{\lambda_j}$, the λ_j perturbed stochastic matrix. The \mathbf{Q} segments facilitate the form of some of the Radon Nykodym derivatives while the perturbed segments come to ensure that $\mu \perp M\{\pi_{\mathbf{Q}}, \mathbf{Q}\}$ and that the ratio set condition is satisfied for cylinder sets.

Notation: By $x = a \pm b$ we mean $a - b \leq x \leq a + b$.

3.1.2. The construction. Choice of the base of induction: Let $M_0 = 1$, $\lambda_1 > 1$, $n_1 = 2$, $N_1 = 3$ and

$$\mathbf{Q}_1 := \begin{pmatrix} \frac{\lambda_1 \varphi}{1 + \lambda_1 \varphi} & 0 & \frac{1}{1 + \lambda_1 \varphi} \\ \frac{\varphi}{1 + \varphi} & 0 & \frac{1}{1 + \varphi} \\ 0 & 1 & 0 \end{pmatrix}$$

be the λ_1 perturbed matrix. Set $P_1 = P_2 = \mathbf{Q}_1$ and $\pi_0 = \pi_{\mathbf{Q}}$. The measures π_1, π_2 are then defined by equation (2.3). Let $m_1 = 3$ and thus $M_1 = 6$. Set $P_j = \mathbf{Q}$ for $j \in [N_1, M_1) = [3, 6)$ and π_3, π_4, π_5 be defined by equation (2.3).

Assume that $\{\lambda_j, m_j, n_j, N_j, M_j\}_{j=1}^{l-1}$ have been chosen.

Choice of λ_l : Notice that the function

$$f(x) := x \frac{1 + \varphi}{1 + \varphi x}.$$

is monotone increasing and continuous in the segment $[1, \infty)$. Therefore we can choose $\lambda_n > 1$ which satisfies the following three conditions:

(1) Finite approximation of the Radon-Nykodym derivatives condition:

$$(3.3) \quad (\lambda_l)^{2m_{l-1}} < e^{\frac{1}{2l}}.$$

This condition ensures an approximation of the derivatives by a finite product.

(2) Lattice condition:

$$(3.4) \quad \lambda_{l-1} \cdot \frac{1+\varphi}{1+\varphi\lambda_{l-1}} \in \left(\lambda_l \cdot \frac{1+\varphi}{1+\varphi\lambda_l} \right)^{\mathbb{N}},$$

where $a^{\mathbb{N}} := \{a^n : n \in \mathbb{N}\}$.

(3) Let

$$\mathbf{Q}_l := \begin{pmatrix} \frac{\varphi\lambda_l}{1+\varphi\lambda_l} & 0 & \frac{1}{1+\varphi\lambda_l} \\ \frac{\varphi}{1+\varphi} & 0 & \frac{1}{1+\varphi} \\ 0 & 1 & 0 \end{pmatrix}.$$

and $\pi_{\mathbf{Q}_l}$ it's unique stationary probability. Notice that when λ_l is close to 1, then \mathbf{Q}_l is close to \mathbf{Q} in the L_∞ sense. Therefore by continuity of the stationary distribution we can demand that

$$(3.5) \quad \|\pi_{\mathbf{Q}} - \pi_{\mathbf{Q}_l}\|_\infty < \frac{1}{2l}.$$

Choice of n_l : It follows from the Lattice condition, equation (3.4), that for each $k \leq l-1$,

$$\left(\lambda_k \cdot \frac{1+\varphi}{1+\varphi\lambda_k} \right) \in \left(\lambda_l \cdot \frac{1+\varphi}{1+\varphi\lambda_l} \right)^{\mathbb{N}}.$$

Choose n_l large enough so that for every $k \leq l-1$ (notice that the demand on $k=1$ is enough) there exists $\mathbb{N} \ni p = p(k, l) \leq \frac{n_l}{20}$ so that

$$(3.6) \quad \left(\lambda_l \cdot \frac{1+\varphi}{1+\varphi\lambda_l} \right)^p = \left(\lambda_k \cdot \frac{1+\varphi}{1+\varphi\lambda_k} \right).$$

Till now we have defined $\{P_j, \pi_j\}_{j=-\infty}^{M_{l-1}}$. By the mean ergodic theorem for Markov chains [LPW, Th. 4.16] and (3.5), one can demand by enlarging n_l if necessary that in addition

$$(3.7) \quad \nu_{\pi_{M_{l-1}}, \mathbf{Q}_l} \left(x : \frac{1}{n_l} \sum_{j=1}^{n_l} \mathbf{1}_{[x_j=1]} \in \left(\frac{1}{\sqrt{5}} - \frac{1}{2l}, \frac{1}{\sqrt{5}} + \frac{1}{2l} \right) \right) > 1 - \frac{1}{l},$$

and

$$(3.8) \quad \nu_{\pi_{M_{l-1}}, \mathbf{Q}_l} \left(x : \frac{1}{n_l} \sum_{j=1}^{n_l} \mathbf{1}_{[x_j=2, x_{j+1}=3]} > \frac{1}{15} \right) > 1 - \frac{1}{l},$$

where ν is the Markov measure on $\{1, 2, 3\}^{\mathbb{N}}$ defined by and \mathbf{Q}_l and $\pi_{M_{l-1}}$. The numbers inside the set were chosen since

$$\pi_{\mathbf{Q}_l}(1) \in \left(\frac{1}{\sqrt{5}} - \frac{1}{2l}, \frac{1}{\sqrt{5}} + \frac{1}{2l} \right),$$

and similarly for l large enough

$$\int \mathbf{1}_{[x_0=2, x_1=3]}(x) d\nu_{\pi_{\mathbf{Q}_l}, \mathbf{Q}_l} = \pi_{\mathbf{Q}_l}(2) (\mathbf{Q}_l)_{2,3} = \left(\frac{1}{\varphi\sqrt{5}} \pm \frac{1}{2l} \right) \frac{1}{\varphi+1} > \frac{1}{15}.$$

Choice of N_l : Let $N_l := M_{l-1} + n_l$. Now set for all $j \in [M_{l-1}, N_l)$,

$$P_j = \mathbf{Q}_l$$

and $\{\pi_j\}_{j=M_{l-1}+1}^{N_l}$ be defined by equation (2.3).

Choice of m_l : Let k_l be the $\left(1 \pm \left(\frac{1}{3}\right)^{3N_l}\right)$ mixing time of \mathbf{Q} . That is for every $\mathbf{n} > k_l$, $A \in \mathcal{F}(0, l)$, $B \in \mathcal{F}(l + \mathbf{n}, \infty)$ and initial distribution $\tilde{\pi}$,

$$(3.9) \quad \nu_{\tilde{\pi}, \mathbf{Q}}(A \cap B) = \left(1 \pm 3^{-3N_l}\right) \nu_{\tilde{\pi}, \mathbf{Q}}(A) \nu_{\pi_{\mathbf{Q}}, \mathbf{Q}}(T^{\mathbf{n}+l}B).$$

Demand in addition that $k_l > N_l$ and

$$(3.10) \quad \left\| \pi_{N_l} \mathbf{Q}^{k_l} - \pi_{\mathbf{Q}} \right\|_{\infty} < 3^{-3N_l}.$$

To explain the last condition notice that equation (2.3) together with the fact that P_j is constant on blocks means that

$$\pi_{N_l} \mathbf{Q}^m = \pi_{N_l+m}.$$

Let m_l be large enough so that

$$(3.11) \quad (1 - 9^{-3N_l})^{m_l/4k_l} \leq \frac{1}{l},$$

and

$$(3.12) \quad (m_l - N_l) \lambda_1^{-2N_l} \geq 1.$$

To summarize the construction. We have defined inductively sequences $\{n_l\}$, $\{N_l\}$, $\{m_l\}$, $\{M_l\}$ of integers which satisfy

$$M_l < N_{l+1} = M_l + n_l < M_{l+1} = N_{l+1} + m_{l+1}.$$

In addition we have defined a monotone decreasing sequence $\{\lambda_l\}$ which decreases to 1 and using that sequence we defined new stochastic matrices $\{\mathbf{Q}_l\}$. Now we set

$$P_j := \begin{cases} \mathbf{Q}, & j \leq 0 \\ \mathbf{Q}_l, & M_{l-1} \leq j < N_l, \\ \mathbf{Q}, & N_l \leq j < M_l \end{cases}$$

and $\pi_j = \pi_{\mathbf{P}}$ for $j \leq 0$. The rest of the π_j 's are defined by the consistency condition, equation (2.3). Finally let μ be the Markovian measure on $\{1, 2, 3\}^{\mathbb{Z}}$ defined by $\{\pi_j, P_j\}_{j=-\infty}^{\infty}$.

Notice that for all $j \in \mathbb{N}$, $\text{supp} P_j \equiv \text{supp} A = \text{supp} \mathbf{Q}$.

3.1.3. Statement of the Theorem and the proof of non singularity and conservativity.

Theorem 6. *The shift $(\{1, 2, 3\}^{\mathbb{Z}}, \mu, T)$ is non singular, conservative, ergodic and of type III_1 .*

Proof. [Non Singularity and K property]

Since $\mu \circ T$ is the markovian measure generated by $\tilde{P}_j = P_{j-1}$ and $\tilde{\pi}_j = \pi_{j-1}$, it follows from (3.1) and the block structure of P_j that the shift is non singular if and only if

$$\sum_{t=1}^{\infty} \sum_{s \in S} \left\{ \left(\sqrt{P_{N_t}(x_{N_t}, s)} - \sqrt{P_{N_{t-1}}(x_{N_t}, s)} \right)^2 + \left(\sqrt{P_{M_t}(x_{M_t}, s)} - \sqrt{P_{M_{t-1}}(x_{M_t}, s)} \right)^2 \right\} < \infty, \quad \mu \circ T \text{ a.s. } x.$$

Since for all $j \in \mathbb{Z}$, $P_j(3, 2) \equiv 1$, $P_j(2, 1) = 1 - P_j(2, 3) \equiv \frac{\varphi}{1+\varphi}$, the sum is dominated by

$$\begin{aligned} \sum_{k=1}^{\infty} \sum_{s \in S} \left\{ 2 \left(\sqrt{P_{N_k}(1, s)} - \sqrt{P_{N_{k-1}}(1, s)} \right)^2 \right\} &= 2 \sum_{k=1}^{\infty} \left\{ \left(\sqrt{\frac{\lambda_k \varphi}{1 + \lambda_k \varphi}} - \sqrt{\frac{\varphi}{1 + \varphi}} \right)^2 \right. \\ &\quad \left. + \left(\sqrt{\frac{1}{1 + \lambda_k \varphi}} - \sqrt{\frac{1}{1 + \varphi}} \right)^2 \right\}. \end{aligned}$$

This sum converges or diverges together with $\sum_{k=1}^{\infty} |\lambda_k - 1|^2$. As a consequence of condition (3.3) on $\{\lambda_j\}$, this sum is finite. Since $P_j \equiv \mathbf{Q}$ for all $j \leq 0$,

$$T'(x) = \frac{d\mu \circ T}{d\mu}(x) = \prod_{k=1}^{\infty} \frac{P_{k-1}(x_k, x_{k+1})}{P_k(x_k, x_{k+1})}.$$

The sequence $\{P_j\}_{j \in \mathbb{Z}}$ satisfies

$$\inf [(P_j P_{j+1} P_{j+2})(s, t) : s, t \in \{1, 2, 3\}, j \in \mathbb{Z}] := c > 0,$$

thus by Proposition 5 the one sided shift $(\{1, 2, 3\}^{\mathbb{N}}, \mathcal{F}, \mu_+, \sigma)$ is an exact factor and T' is \mathcal{F} measurable and thus the shift is a K automorphism. Here μ_+ denotes the measure on the one sided shift space defined by $\{\pi_j, P_j\}_{j \geq 1}$. \square

In order to show the other properties of the Markov Shift, we will need a more concrete expression of the Radon Nykodym derivatives. The measure μ , or more concretely it's transition matrices, differs from the stationary $\{\pi_{\mathbf{Q}}, \mathbf{Q}\}$ measure only when one moves inside state 1 in the segments $[M_j, N_{j+1})$. Denote by

$$L_j(x) := \# \{k \in [M_{j-1}, N_j) : x_k = 1\}$$

and

$$V_j(x) = \# \{k \in [M_{j-1}, N_j) : x_k = x_{k+1} = 1\}.$$

Lemma 7. *For every $\epsilon > 0$, there exists $t_0 \in \mathbb{N}$ s.t for every $t > t_0$, $N_t \leq n < m_t$ and $x \in \{1, 2, 3\}^{\mathbb{Z}}$,*

$$(T^n)'(x) = (1 \pm \epsilon) \prod_{k=1}^t \left[\left(\frac{1 + \varphi}{1 + \varphi \lambda_k} \right)^{L_k \circ T^n(x) - L_k(x)} \cdot \lambda_k^{V_k \circ T^n(x) - V_k(x)} \right].$$

Proof. Let $\epsilon > 0$, $t \in \mathbb{N}$ and $N_t \leq n < m_t$. Canceling out all the k 's such that $P_{k-n} = P_k$ one can see that

$$(T^n)'(x) = I_t \cdot \tilde{I}_t$$

where

$$I_t = \prod_{u=1}^t \left[\left(\prod_{k=M_{u-1}}^{N_u} \frac{P_{k-n}(x_k, x_{k+1})}{P_k(x_k, x_{k+1})} \right) \cdot \left(\prod_{k=M_{u-1}+n}^{N_u+n-1} \frac{P_{k-n}(x_k, x_{k+1})}{P_k(x_k, x_{k+1})} \right) \right]$$

and

$$\tilde{I}_t = \prod_{u=t+1}^{\infty} \left[\left(\prod_{k=M_{u-1}}^{M_{u-1}+n-1} \frac{P_{k-n}(x_k, x_{k+1})}{P_k(x_k, x_{k+1})} \right) \cdot \left(\prod_{k=N_u}^{N_u+n-1} \frac{P_{k-n}(x_k, x_{k+1})}{P_k(x_k, x_{k+1})} \right) \right].$$

We will analyze the two terms separately. Since for every $M_{u-1} \leq k < M_{u-1} + n$, $P_k = \mathbf{Q}_u$ and $P_{k-n} = \mathbf{Q}$,

$$\frac{P_{k-n}(x_k, x_{k+1})}{P_k(x_k, x_{k+1})} \leq \frac{\mathbf{Q}_{1,3}}{(\mathbf{Q}_u)_{1,3}} = \frac{1 + \varphi \lambda_u}{1 + \varphi} \leq \lambda_u.$$

Similarly for $N_u \leq k < N_u + n$, $P^{(k)} = \mathbf{Q}$ and $P_{k-n} = \mathbf{Q}_u$. Therefore

$$\frac{P_{k-n}(x_k, x_{k+1})}{P_k(x_k, x_{k+1})} \leq \frac{(\mathbf{Q}_u)_{1,1}}{\mathbf{Q}_{1,1}} \leq \lambda_u.$$

and

$$\lambda_u^{-2n} \leq \left(\prod_{k=M_{u-1}}^{M_{u-1}+n-1} \frac{P_{k-n}(x_k, x_{k+1})}{P_k(x_k, x_{k+1})} \right) \cdot \left(\prod_{k=N_u}^{N_u+n-1} \frac{P_{k-n}(x_k, x_{k+1})}{P_k(x_k, x_{k+1})} \right) \leq \lambda_u^{2n},$$

here the lower bound is achieved by a similar analysis. This gives

$$\begin{aligned}
\tilde{I}_t &= \prod_{u=t+1}^{\infty} [\lambda_u^{\pm 2n}] \\
&= \prod_{u=t+1}^{\infty} [\lambda_u^{\pm 2m_{u-1}}] \quad (\text{since } \forall u > t, n < m_t < m_u) \\
&\stackrel{(3.3)}{=} e^{\pm \sum_{n=t+1}^{\infty} \frac{1}{2^n}} \xrightarrow{t \rightarrow \infty} 1.
\end{aligned}$$

Consequently there exists $t_0 \in \mathbb{N}$ so that for all $x \in \Sigma_{\mathbf{A}}$, for all $t > t_0$ and $N_t \leq n \leq m_t$,

$$(T^n)'(x) = (1 \pm \epsilon)I_t.$$

By noticing that for $k \in \bigcup_{j=1}^t ([M_{j-1}, N_j] \cup [M_{j-1} + n, N_j + n])$,

$$P_{k-n}(x_k, x_{k+1}) \neq P_k(x_k, x_{k+1})$$

if and only if $x_k = 1$ one can check that

$$I_t = \prod_{k=1}^t \left[\left(\frac{1 + \varphi}{1 + \varphi \lambda_k} \right)^{L_k \circ T^n(x) - L_k(x)} \lambda_k^{V_k \circ T^n(x) - V_k(x)} \right].$$

□

Corollary 8. *The shift $(\{1, 2, 3\}^{\mathbb{Z}}, \mu, T)$ is conservative and ergodic.*

Proof. Since the shift is a K -automorphism it is enough to prove conservativity.

For every $j \in \mathbb{N}$, $0 \leq L_k(x), D_k(x) \leq n_k$. Whence

$$\begin{aligned}
\left(\frac{1 + \varphi}{1 + \varphi \lambda_k} \right)^{L_k \circ T^n(x) - L_k(x)} \lambda_k^{V_k \circ T^n(x) - V_k(x)} &\geq \left(\frac{1 + \varphi}{1 + \varphi \lambda_k} \right)^{L_k \circ T^n(x)} \lambda_k^{-V_k(x)} \\
&\geq \lambda_k^{-2n_k} \geq \lambda_1^{-2n_k},
\end{aligned}$$

and for every $t \in \mathbb{N}$,

$$\prod_{k=1}^t \left[\left(\frac{1 + \varphi}{1 + \varphi \lambda_k} \right)^{L_k \circ T^n(x) - L_k(x)} \lambda_k^{V_k \circ T^n(x) - V_k(x)} \right] \geq \lambda_1^{-2 \sum_{k=1}^t n_k} \geq \lambda_1^{-2N_t}.$$

By Lemma 7 there exists $t_0 \in \mathbb{N}$ such that for all $t \geq t_0$, $N_t \leq n \leq m_t$ and $x \in \Sigma_{\mathbf{A}}$, $(T^n)'(x) \geq \frac{\lambda_1^{-2N_t}}{2}$. Therefore for all $x \in \Sigma_{\mathbf{A}}$,

$$\sum_{n=1}^{\infty} (T^n)'(x) \geq \sum_{t=1}^{\infty} \sum_{n=N_t}^{m_t} (T^n)'(x) \geq \sum_{t=t_0}^{\infty} \frac{1}{2} (m_t - N_t) \lambda_1^{-2N_t} \stackrel{(3.12)}{=} \infty.$$

By Hopf's criteria the shift is conservative. □

3.1.4. Proof of the type III₁ property. In order to prove that the ratio set is $[0, \infty)$ we are going to use the following principle: since $R(T)$ is a multiplicative subset it is enough to show that there exists $y_n \in R(T) \setminus \{1\}$ with $y_n \rightarrow 1$ as $n \rightarrow \infty$.

Theorem 9. *Let μ be the Markovian measure constructed in Subsection 3.1.2. For every $n \in \mathbb{N}$, $\lambda_n \cdot \frac{1+\varphi}{1+\varphi\lambda_n} \in R(T)$ and therefore the shift is type III₁.*

Fix $n \in \mathbb{N}$. The first stage in proving that $\lambda_n \cdot \frac{1+\varphi}{1+\varphi\lambda_n} \in R(T)$ is to show that the ratio set condition is satisfied for all cylinders with a positive proportion of the measure of the cylinder set. Then for a general $A \in \mathcal{B}_+$, we use the density of cylinder sets in \mathcal{B} .

Given $t \in \mathbb{N}$, denote by $\mathcal{C}(t)$ the collection of all $[c]_0^{N_t}$ cylinder sets such that

$$(3.13) \quad L_t(c) = \sum_{k=M_{t-1}}^{N_t-1} \mathbf{1}_{[c_k=1]} \in \left(\frac{n_t}{4}, \frac{n_t}{2}\right) \text{ and } \sum_{k=M_{t-1}}^{N_t-1} \mathbf{1}_{[c_k=2, c_{k+1}=3]} \geq \frac{n_t}{15}.$$

Since

$$\mu\left([c]_{M_{t-1}}^{N_t}\right) = \nu_{\pi_{M_{t-1}}, \mathbf{Q}_t}([c]_0^{n_t}),$$

it follows from (3.7) and (3.8) that for all t large enough,

$$\mu\left(\bigcup_{C \in \mathcal{C}(t)} C\right) \geq 1 - \frac{1}{2t}.$$

In order to shorten the notation, given $M, j \in \mathbb{N}$, $B \in \mathcal{B}$ and $\epsilon > 0$, let

$$\mathfrak{RSC}(M, B, j, \epsilon) := B \cap T^{-M} B \cap \left[(T^M)' = \lambda_j \cdot \frac{1 + \varphi}{1 + \varphi \lambda_j} \cdot (1 \pm \epsilon)\right],$$

and for $M \in \mathbb{N}$,

$$\Sigma_{\mathbf{A}}(M) := \{1, 2, 3\}^M \cap \Sigma_{\mathbf{A}}.$$

Lemma 10. *For every $[b]_{-n}^n$ cylinder set, $\epsilon > 0$ and $j \in \mathbb{N}$, there exists a $t_0 \in \mathbb{N}$ so that for all $t > t_0$ the following holds:*

For every $C = [c]_0^{N_t-1} \in \mathcal{C}(t)$ there exists $d = d(b, C) \in \Sigma_{\mathbf{A}}(N_t + n)$ such that for every $\mathbb{N} \ni l \leq m_t/k_t$,

$$(3.14) \quad C \cap [d]_{lk_t-n}^{lk_t+N_t-1} \subset T^{-lk_t} [b]_{-n}^n \cap \left[(T^{lk_t})' = \lambda_j \cdot \frac{1 + \varphi}{1 + \varphi \lambda_j} \cdot (1 \pm \epsilon)\right].$$

Recall that $k_t > N_t$ is defined as a (1 ± 3^{-3N_t}) mixing time for \mathbf{Q} .

Proof. Let $[b]_{-n}^n, \epsilon > 0$ and $j \in \mathbb{N}$ be given. By Lemma 7 there exists τ such that for every $t \geq \tau$ and $1 \leq l \leq m_t/k_t$ (here $lk_t \in [N_t, m_t]$),

$$\left(T^{lk_t}\right)'(x) = (1 \pm \epsilon) \prod_{k=1}^t \left[\left(\frac{1 + \varphi}{1 + \varphi \lambda_k} \right)^{L_k \circ T^{lk_t}(x) - L_k(x)} \cdot \lambda_k^{V_k \circ T^{lk_t}(x) - V_k(x)} \right].$$

Choose t_0 to be any integer which satisfies $t_0 > \max(\tau, j)$ and $M_{t_0} > n$.

Let $t > t_0$ and choose a cylinder set $[c]_0^{N_t} \in \mathcal{C}(t)$ which intersects $[b]_{-n}^n$. That is $c_i = b_i$ for $i \in [0, n]$. We need now to choose $d \in \Sigma_{\mathbf{A}}(N_t + n)$ which satisfies (3.14). Notice that for $x \in [d]_{lk_t-n}^{lk_t+N_t} \cap [c]_0^{N_t}$,

$$\begin{aligned} & \prod_{k=1}^t \left[\left(\frac{1 + \varphi}{1 + \varphi \lambda_k} \right)^{L_k \circ T^{lk_t}(x) - L_k(x)} \cdot \lambda_k^{V_k \circ T^{lk_t}(x) - V_k(x)} \right] \\ &= \prod_{k=1}^t \left[\left(\frac{1 + \varphi}{1 + \varphi \lambda_k} \right)^{L_k(d) - L_k(c)} \cdot \lambda_k^{V_k(d) - V_k(c)} \right], \end{aligned}$$

in this representation we look at $[d]_{-n}^{N_t}$. For all $k \in [0, M_{t-1}]$, let

$$d_k = c_k$$

and for all $k \in [-n, 0)$,

$$d_k = b_k.$$

Notice that this means that for $k \in [-n, n]$, $d_k = b_k$ and thus

$$[d]_{lk_t-n}^{lk_t+N_t} \subset T^{-lk_t} [b]_{-n}^n.$$

Let $p(j, t) \leq \frac{n_t}{20}$ be the integer (condition (3.6)) such that

$$\left(\lambda_t \cdot \frac{1 + \varphi}{1 + \varphi \lambda_t} \right)^{p(j, t)} = \lambda_j \cdot \frac{1 + \varphi}{1 + \varphi \lambda_j}.$$

Set $d_k = 1$ for all $k \in [M_{t-1}, M_{t-1} + V_t(c) + p(j, t)]$ and then continue repeatedly with the sequence "321" $L_t(c) - V_t(c)$ times. Since c satisfies (3.13), this construction is well defined (e.g. we have not reached yet $k = N_t - 1$). Continue with sequences of 32 till $k = N_t - 1$.

Thus we have defined d in such a way so that

$$L_t(d) - L_t(c) = p(j, t)$$

and

$$V_t(d) - V_t(c) = p(j, t).$$

In addition for all $0 \leq k < t$,

$$L_k(d) = L_k(c) \text{ and } V_k(c) = V_k(d).$$

Thus for all $x \in [d]_{lk_t - n}^{lk_t + N_t} \cap [c]_0^{N_t}$,

$$\begin{aligned} \left(T^{lk_t} \right)'(x) &= (1 \pm \epsilon) \prod_{k=1}^t \left[\left(\frac{1 + \varphi}{1 + \varphi \lambda_k} \right)^{L_k(d) - L_k(c)} \cdot \lambda_k^{V_k(d) - V_k(c)} \right] \\ &= (1 \pm \epsilon) \left(\lambda_t \cdot \frac{1 + \varphi}{1 + \varphi \lambda_t} \right)^{p(j, t)} \\ &= (1 \pm \epsilon) \left(\lambda_j \cdot \frac{1 + \varphi}{1 + \varphi \lambda_j} \right). \end{aligned}$$

This proves the lemma. □

In the course of the proof one sees that the event

$$\left([b]_{-n}^n \cap [c]_0^{N_t} \right) \cap \left(T^{-lk_t} [b]_{-n}^n \cap \left\{ \left(T^{lk_t} \right)' = \lambda_j \cdot \frac{1 + \varphi}{1 + \varphi \lambda_j} \cdot (1 \pm \epsilon) \right\} \right)^c$$

is $\mathcal{F}(lk_t - n, lk_t + N_t)$ measurable and does not depend on $\mathcal{F}(lk_t + N_t, lk_t + 2N_t)$.

Remark 11. Given $[c]_0^{N_t} \in \mathcal{C}_t$ we have defined $d = d(c) \in \Sigma_{\mathbf{A}}(N_t + n)$. The definition of d is not necessarily one to one. This is because if $[\tilde{c}]_0^{M_{t-1}} = [c]_0^{M_{t-1}}$, $V_t(c) = V_t(\tilde{c})$ and $L_t(c) = L_t(\tilde{c})$ then $d(c) = d(\tilde{c})$. In order to make it one to one we will use

$$[d(c), c]_{lk_t - n}^{lk_t + 2N_t}$$

instead of $[d(c)]_{lk_t}^{lk_t + N_t}$ where by $[a, b]_l^{l + \text{length}(a) + \text{length}(b)}$ we mean the concatenation of a and b . This can be thought of as putting a Marker on $d(c)$. In order that the concatenation will be in Σ_A we need that

$$\mathbf{Q}(d(c)_{N_t - 1}, c_0) > 0.$$

This can be done by possibly changing the last two coordinates of $d(c)$. This will change the value of $\left(T^{lk_t} \right)'$ by at most a factor of $\lambda_t^{\pm 4}$, which is close enough to one. We will denote by $\mathbf{d}(c) := (\tilde{d}(c), c)$. We still have

$$[\mathbf{d}(c)]_{lk_t - n}^{lk_t + 2N_t} \subset T^{-lk_t} [b]_{-n}^n \cap \left[\left(T^{lk_t} \right)' = \lambda_j \cdot \frac{1 + \varphi}{1 + \varphi \lambda_j} \cdot (1 \pm \epsilon) \right],$$

but now the map $c \mapsto \mathbf{d}(c)$ is one to one.

In the proof of the next lemma we will make use of the fact that for every cylinder set $([a]_m^l)^c$ is $\mathcal{F}(m, l)$ measurable.

Lemma 12. For every $[b]_{-n}^n$ cylinder set, $\epsilon > 0$ and $j \in \mathbb{N}$ there exists $t_0 \in \mathbb{N}$ such that for all $t > t_0$,

$$\mu \left(\bigcup_{l=1}^{m_t/4k_t} \mathfrak{RSE}(4lk_t, [b]_{-n}^n, j, \epsilon) \right) \geq 0.8\mu([b]_{-n}^n).$$

Proof. Let $[b]_{-n}^n$ be a cylinder set and t_0 be as in Lemma 10. For all $t \geq t_0$, $[c]_0^{N_t} \in \mathcal{C}(t)$ which intersects $[b]_{-n}^n$ and $1 \leq l \leq m_t/4k_t$,

$$\left([c]_0^{N_t} \cap [b]_{-n}^n \right) \cap \left(\mathfrak{RSE}(4lk_t, [b]_{-n}^n, j, \epsilon) \right)^c \subset [c]_0^{N_t} \cap [b]_{-n}^n \cap \left([d(c)]_{4lk_t-n}^{4lk_t+N_t} \right)^c$$

The fact that $P_j \equiv \mathbf{Q}$ for $j \in [N_t, M_t]$ implies that

$$\begin{aligned} \mu \left([d(c)]_{4lk_t-n}^{4lk_t+2N_t} \right) &= \nu_{\pi_{4lk_t-n}, \mathbf{Q}} \left([d(c)]_0^{4lk_t+2N_t+n} \right) \\ &\geq \pi_{4lk_t-n}(d_{-n}) \mathbf{Q}_{1,3}^{2N_t+n-1} \\ &\geq \left(\pi_{\mathbf{Q}}(d_{-n}) - \frac{1}{3^{N_t}} \right) \mathbf{Q}_{1,3}^{2N_t+n-1} \quad (\text{by (3.10)}) \\ &\gtrsim \frac{1}{3^{3N_t}}. \end{aligned}$$

Therefore, one has by many application of (3.9) (mixing time condition),

$$\begin{aligned} &\mu \left(\left([b]_{-n}^n \cap [c]_0^{N_t} \right) \cap \left(\bigcup_{l=1}^{m_t/4k_t} \mathfrak{RSE}(lk_t, [b]_{-n}^n, j, \epsilon) \right)^c \right) \\ &\leq \mu \left(\left([b]_{-n}^n \cap [c]_0^{N_t} \right) \cap \left\{ \bigcap_{l=1}^{m_t/4k_t} \left([d_c]_{4lk_t}^{4lk_t+N_t} \right)^c \right\} \right) \\ &\leq \mu \left(\left([b]_{-n}^n \cap [c]_0^{N_t} \right) \right) \prod_{l=1}^{m_t/4k_t} \left[(1 + 3^{-3N_t}) \left(1 - \nu_{\pi_{\mathbf{Q}}, \mathbf{Q}} \left([d_c]_{4lk_t}^{4lk_t+N_t} \right) \right) \right] \\ &\leq \mu \left(\left([b]_{-n}^n \cap [c]_0^{N_t} \right) \right) \left[(1 + 3^{-3N_t}) (1 - 3^{-3N_t}) \right]^{m_t/4k_t} \\ &\stackrel{(3.11)}{\leq} \frac{1}{t} \mu \left(\left([b]_{-n}^n \cap [c]_0^{N_t} \right) \right). \end{aligned}$$

Notice that we used the fact that

$$(4(l+1)k_t - n) - (4lk_t + 2N_t) > (4l+3)k_t - (4l+2)k_t = k_t.$$

If t is large enough then

$$\mu \left(\Sigma_{\mathbf{A}} \setminus \bigcup_{C \in \mathcal{C}(t)} C \right) < 0.1\mu([b]_{-n}^n),$$

and for all $[c]_0^{N_t} = C \in \mathcal{C}(t)$,

$$\begin{aligned} \mu \left([b]_{-n}^n \cap [c]_0^{N_t} \cap \left(\bigcup_{l=1}^{m_t/4k_t} \mathfrak{RSE}(lk_t, [b]_{-n}^n, j, \epsilon) \right) \right) &> \left(1 - \frac{1}{t} \right) \mu([b]_{-n}^n \cap [c]_0^{N_t}) \\ &\geq 0.9\mu([b]_{-n}^n \cap [c]_0^{N_t}). \end{aligned}$$

The Lemma follows from

$$\begin{aligned}
& \mu \left(\bigcup_{l=1}^{m_t/4k_t} \mathfrak{RSE}(lk_t, [b]_{-n}^n, j, \epsilon) \right) \\
& \geq \mu \left(\biguplus_{[c]_0^{N_t} \in \mathcal{C}(t)} [b]_{-n}^n \cap [c]_0^{N_t} \cap \bigcup_{l=1}^{m_t/4k_t} \mathfrak{RSE}(lk_t, [b]_{-n}^n, j, \epsilon) \right) \\
& \geq 0.9 \sum_{[c]_0^{N_t} \in \mathcal{C}(t)} \mu([b]_{-n}^n \cap [c]_0^{N_t}) \\
& \geq 0.8\mu([b]_{-n}^n)
\end{aligned}$$

□

Proof of Theorem 9. This is a standard approximation technique. Let $j \in \mathbb{N}$, $A \in \mathcal{B}$, $\mu(A) > 0$ and $\epsilon > 0$. Since the ratio set condition on the derivative is monotone with respect to ϵ and

$$1 < \lambda_j \cdot \frac{1 + \varphi}{1 + \varphi \lambda_j} < 2,$$

we can assume that

$$(3.15) \quad 1 \leq \lambda_j \cdot \frac{1 + \varphi}{1 + \varphi \lambda_j} (1 \pm \epsilon) \leq 2.$$

Since $\mathcal{F}(-n, n) \uparrow \mathcal{B}$ as $n \rightarrow \infty$, there exists a cylinder set $\mathfrak{b} = [b]_{-n}^n$ such that

$$\mu(A \cap \mathfrak{b}) > 0.99\mu(\mathfrak{b}).$$

By Lemma 12 there exists $t \in \mathbb{N}$ for which

$$\mu \left(\mathfrak{b} \cap \left\{ \bigcup_{l=1}^{m_t/4k_t} T^{-4lk_t} \mathfrak{b} \cap \left[\left(T^{4lk_t} \right)' = \lambda_j \cdot \frac{1 + \varphi}{1 + \varphi \lambda_j} \cdot (1 \pm \epsilon) \right] \right\} \right) > 0.8\mu(\mathfrak{b}).$$

Denote by

$$B = \mathfrak{b} \cap \left\{ \bigcup_{l=1}^{m_t/4k_t} T^{-4lk_t} \mathfrak{b} \cap \left[\left(T^{4lk_t} \right)' = \lambda_j \cdot \frac{1 + \varphi}{1 + \varphi \lambda_j} \cdot (1 \pm \epsilon) \right] \right\}.$$

We can assume that for $x \in B$, there exists $C(x) = [c]_0^{N_t} \in \mathcal{C}_t$ so that $x \in C(x)$. Then by the proof of Lemma 10 there exists $d(C(x)) \in \Sigma_{\mathbf{A}}(2N_t + n)$ such that if $x \in [d(C(x))]_{4lk_t - n}^{4lk_t + 2N_t}$, then

$$(3.16) \quad \left(T^{4lk_t} \right)'(x) = \lambda_j \cdot \frac{1 + \varphi}{1 + \varphi \lambda_j} \cdot (1 \pm \epsilon) \text{ and } x \in T^{-4lk_t} \mathfrak{b}.$$

Define $\phi : B \rightarrow \mathbb{N}$

$$\phi(x) := \inf \left\{ l \leq m_t/4k_t : [x]_{4lk_t - n}^{4lk_t + 2N_t} = [\mathbf{d}(C(x))]_{4lk_t - n}^{4lk_t + 2N_t} \right\}$$

and $S = T^\phi : B \rightarrow S(B) \subset \mathfrak{b}$. We claim that S is one to one. Indeed, since the map $[c]_0^{N_t} \mapsto \mathbf{d}(c)$ is one to one, for every $x, y \in B$ such that $C(x) \neq C(y)$,

$$[Sy]_{-n}^{2N_t} = [\mathbf{d}(C(y))]_{-n}^{2N_t} \neq [\mathbf{d}(C(x))]_{-n}^{2N_t} = [Sx]_{-n}^{2N_t},$$

consequently $Sx \neq Sy$. In addition, by the definition of ϕ , if $x \neq y$ and $C(x) = C(y)$ then $Sx \neq Sy$.

It follows from (3.16) and (3.15), that for all $x \in B$,

$$S'(x) := \frac{d\mu \circ S}{d\mu}(x) = \lambda_j \cdot \frac{1 + \varphi}{1 + \varphi \lambda_j} \cdot (1 \pm \epsilon) \in [1, 2].$$

Therefore $\frac{d\mu \circ S^{-1}}{d\mu}(y) \geq \frac{1}{2}$ for all $y \in S(B)$. A calculation shows that

$$\begin{aligned} \mu(S(B) \cap A) &> \mu(S(B)) - \mu(\mathfrak{b} \setminus A) \\ &> \mu(B) - \mu(\mathfrak{b} \setminus A) \\ &= 0.79\mu(\mathfrak{b}), \end{aligned}$$

and

$$\mu(S^{-1}(S(B) \cap A)) > \frac{\mu(S(B) \cap A)}{2} > 0.39\mu(\mathfrak{b}).$$

So

$$\begin{aligned} &\sum_{l=1}^{m_t/4lk_t} \mu \left(A \cap \left\{ T^{-4lk_t} A \cap \left[\left(T^{4lk_t} \right)' = \lambda_j \cdot \frac{1+\varphi}{1+\varphi\lambda_j} \cdot (1 \pm \epsilon) \right] \right\} \cap [\phi = 4lk_t] \right) \\ &\geq \mu((B \cap A) \cap S^{-1}(S(B) \cap A)) \quad \{\text{Notice that } B, S(B) \subset \mathfrak{b}\} \\ &\geq \mu(B \cap A) - \mu(\mathfrak{b} \setminus S^{-1}(S(B) \cap A)) \\ &\geq 0.18\mu(\mathfrak{b}), \end{aligned}$$

and thus there exists $l \in \mathbb{N}$ such that

$$\mu \left(A \cap T^{-4lk_t} A \cap \left[\left(T^{4lk_t} \right)' = \lambda_j \cdot \frac{1+\varphi}{1+\varphi\lambda_j} \cdot (1 \pm \epsilon) \right] \right) > 0.$$

This proves the Theorem. □

4. CONCLUDING REMARKS

One feature of this construction is that if $f(x, y) = (x + y, x)$, $\Phi : \Sigma_A \rightarrow \mathbb{T}^2$ is the semi conjugacy map constructed from the Markov partition $\{R_1, R_2, R_3\}$ and $\nu = \mu \circ \Phi^{-1}$, then:

- $\Phi : (\Sigma_A, \mu, T) \rightarrow (\mathbb{T}^2, \nu, f)$ is one to one, hence a measure theoretic isomorphism.

For every $n_k < n_{k-1} < \dots < n_1 < 0$ and $i_1, \dots, i_k \in \{1, 2, 3\}$,

$$\nu \left(\bigcap_{j=1}^k f^{-n_j} R_{i_k} \right) = \text{Leb} \left(\bigcap_{j=1}^k f^{-n_j} R_{i_k} \right).$$

Remark 13. Given a mixing TMS Σ_A , one can construct a type III₁ Markov shift supported on Σ_A as follows. Define Q to be the matrix with

$$Q_{i,j} = \begin{cases} 1/\sum_{l=1}^n A_{i,l}, & A_{i,j} = 1 \\ 0 & A_{i,j} = 0 \end{cases}.$$

If A has a row $i \in \{1, \dots, |S|\}$ with at least two 1's, one can proceed as in our example to define Q_l to be the matrix Q perturbed in the i -th row between two non zero coordinates. The rest of the proof remains the same.

We end this section with the following open question: Given a mixing TMS $\Sigma \subset F^{\mathbb{Z}}$ with F finite. Does there exist a Markov measure $\nu = M\{\pi_k, R_k\}$ so that:

- $\nu \circ T \sim \nu$.
- ν is fully supported on Σ and the shift is ergodic with respect to ν .
- The non singular Markov shift $(\Sigma, \mathcal{B}, \nu, T)$ is of type II_∞ (preserves an a.c.i.m but no a.c.i.p.) or III_λ, $0 \leq \lambda < 1$.
- Preferably ν is half stationary meaning that there exists an irreducible and aperiodic stochastic Matrix R so that for all $k < 0$, $R_k = R$ and $\pi_k = \pi_R$.

If in addition Σ is a *TMS* arising from a hyperbolic Toral automorphism and R is the stochastic matrix representing the Lebesgue measure then a positive answer to this question may give new examples of Anosov diffeomorphisms by the methods of [Kos1]. In the case where $\Sigma = \{0, 1\}^{\mathbb{Z}}$ and ν is a half stationary product measure it was shown in [Kos2] that the shift is either of type III_1 , dissipative or equivalent to a classical Bernoulli shift (ν is equivalent to a product measure with i.i.d entries).

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